Problem 1.4

Direction cosines

The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosines of the angles between the vector and the x, y, and z axes are usually called, in turn, α, β, and γ. Prove that $\alpha^2 + \beta^2 + \gamma^2 = 1$, using either geometry or vector algebra.

Solution

Suppose we have a vector,

$$\mathbf{F} = \langle F_x, F_y, F_z \rangle.$$

Let the angle between $\mathbf{F}$ and the x-axis be θ_x, the angle between $\mathbf{F}$ and the y-axis be θ_y, and the angle between $\mathbf{F}$ and the z-axis be θ_z.

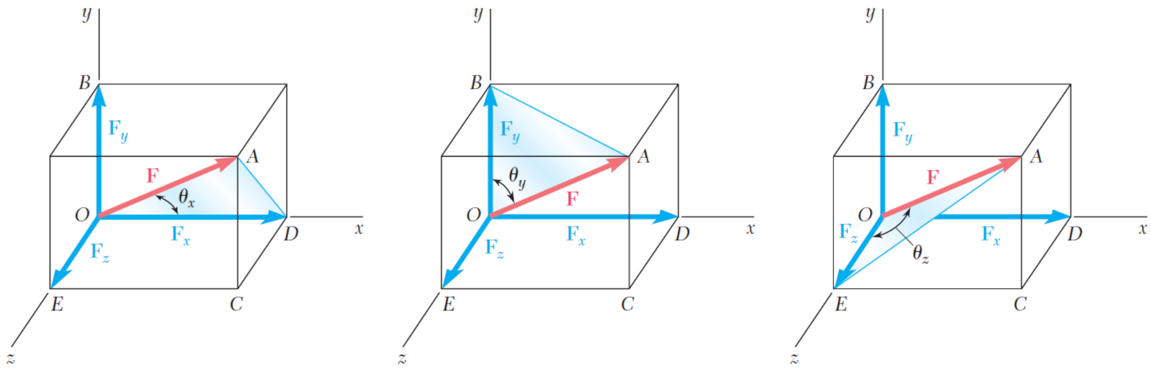

Figure 1: An illustration of the angles between a vector and the coordinate axes.

Each of the shaded areas in the figure are right triangles. Take the cosine of each angle.

$$\cos\theta_x = \frac{F_x}{|\mathbf{F}|} = \frac{F_x}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$$

$$\cos\theta_y = \frac{F_y}{|\mathbf{F}|} = \frac{F_y}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$$

$$\cos\theta_z = \frac{F_z}{|\mathbf{F}|} = \frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$$

Hence,

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = \frac{F_x^2}{F_x^2 + F_y^2 + F_z^2} + \frac{F_y^2}{F_x^2 + F_y^2 + F_z^2} + \frac{F_z^2}{F_x^2 + F_y^2 + F_z^2} = 1.$$

And since

$$\alpha = \cos\theta_x$$
$$\beta = \cos\theta_y$$
$$\gamma = \cos\theta_z,$$

we obtain the desired result.

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$